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Robert E. Megginson

An Introduction to Banach Space Theory

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To my mother and father
and, of course, to Kathy

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*An asterisk preceding a section number indicates an optional section.

Preface

A *normed space* is a real or complex vector space X along with a *norm function* on the space; that is, a function $\|\cdot\|$ from X into the nonnegative reals such that if $x, y \in X$ and α is a scalar, then $\|\alpha x\| = |\alpha|\,\|x\|$, $\|x+y\| \leq \|x\| + \|y\|$, and $\|x\| = 0$ precisely when $x = 0$. A *Banach space* is then defined to be a normed space such that the metric given by the formula $d(x, y) = \|x - y\|$ is complete. In a sense, the study of Banach spaces is as old as the study of the properties of the absolute value function on the real numbers. However, the general theory of normed spaces and Banach spaces is a much more recent development. It was not until 1904 that Maurice René Frechet [80] suggested that it might be fruitful to extend the notion of limit from specific situations commonly studied in analysis to a more general setting. In his 1906 thesis [81], he developed the notion of a general metric space and immediately embarked on the study of real $C[a, b]$, the vector space of all real-valued continuous functions on a compact interval $[a, b]$ of the real line with the metric given by the formula $d(f, g) = \max\{\, |f(t) - g(t)| : t \in [a, b]\,\}$. In this seminal work on metric space theory, Fréchet was already emphasizing the important role played by the completeness of metrics such as that of $C[a, b]$. He was also doing a bit of Banach space theory since his metric for $C[a, b]$ is induced by a norm, as will be seen in Example 1.2.10 of this book. By 1908, Erhard Schmidt [211] was using the modern notation $\|x\|$ for the norm of an element x of ℓ_2, but the formulation of the general definition of a normed space had to wait a few more years.

Since the study of normed spaces for their own sake evolved rather than arose fully formed, there is some room to disagree about who founded the field. Albert Bennett came close to giving the definition of a normed space in a 1916 paper [23] on an extension of Newton's method for finding roots, and in 1918 Frédéric Riesz [195] based a generalization of the Fredholm theory of integral equations on the defining axioms of a complete normed space, though he did not use these axioms to study the general theory of such spaces. According to Jean Dieudonné [64], Riesz had at this time considered developing a general theory of complete normed spaces, but never published anything in this direction. In a paper that appeared in 1921, Eduard Helly [102] proved what is now called Helly's theorem for bounded linear functionals. Along the way, he developed some of the general theory of normed spaces, but only in the context of norms on subspaces of the vector space of all sequences of complex scalars.

The first undisputed efforts to develop the general theory of normed spaces appeared independently in a paper by Hans Hahn [98] and in Stefan Banach's thesis [10], both published in 1922. Both treatments considered only complete normed spaces. Though the growth of the general theory proceeded through the 1920s, the real impetus for the development of modern Banach space theory was the appearance in 1932 of Banach's book *Théorie des Opérations Linéaires* [13], which stood for years as the standard reference work in the field and is still profitable reading for the Banach space specialist today.

Many important reference works in the field have appeared since Banach's book, including, among others, those by Mahlon Day [56] and by Joram Lindenstrauss and Lior Tzafriri [156, 157]. While those works are classical starting points for the graduate student wishing to do research in Banach space theory, they can be formidable reading for the student who has just completed a course in measure theory, found the theory of L_p spaces fascinating, and would like to know more about Banach spaces in general.

The purpose of this book is to bridge that gap. Specifically, this book is for the student who has had enough analysis and measure theory to know the basic properties of the L_p spaces, and is designed to prepare such a student to read the type of work mentioned above as well as some of the current research in Banach space theory. In one sense, that makes this book a functional analysis text, and in fact many of the classical results of functional analysis are in here. However, those results will be applied almost exclusively to normed spaces in general and Banach spaces in particular, allowing a much more extensive development of that theory while placing correspondingly less emphasis on other topics that would appear in a traditional functional analysis text.

It should be made clear that this book is an introduction to the general theory of Banach spaces, not a detailed survey of the structure of the classical Banach spaces. Along the way, the reader will learn quite a bit about the classical Banach spaces from their extensive use in the theory, examples, and exercises. Those who find their appetite for those spaces whetted have an entire feast awaiting them in the volumes of Lindenstrauss and Tzafriri.

Prerequisites

Appendix A contains a detailed list of the prerequisites for reading this book. Actually, these prerequisites can be summarized very briefly: Anyone who has studied the first third of Walter Rudin's *Real and Complex Analysis* [202], which is to say the first six chapters of that book, will be able to read this book through, cover-to-cover, omitting nothing. Of course, this implies that the reader has had the basic grounding in undergraduate mathematics necessary to tackle Rudin's book, which should include a first course in linear algebra. Though some knowledge of elementary topology beyond the theory of metric spaces is assumed, the topology presented near the beginning of Rudin's book is enough. In short, all of this book is accessible to someone who has had a course in real and complex analysis that includes the duality between the Lebesgue spaces L_p and L_q when $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, as well as the Riesz representation theorem for bounded linear functionals on $C(K)$ where K is a compact Hausdorff space, and who has not slighted the usual prerequisites for such a course.

In fact, a large amount of this book is accessible at a much earlier stage in a student's mathematical career. The real reason for the measure theory prerequisite is to allow the reader to see applications of Banach space theory to the L_p spaces and spaces of measures, not because the measure theory is itself crucial to the development of the general Banach space theory in this book. It is quite possible to use this book as the basis for an undergraduate topics course in Banach space theory that concentrates on the metric theory of finite-dimensional Banach spaces and the spaces ℓ_p and c_0, for which the only prerequisites are a first course in linear algebra, a first course in real analysis without measure theory, and an introduction to metric spaces without the more general theory of topological spaces. Appendix A explains in detail how to do so. A list of the properties of metric spaces with which a student in such a course should be familiar can be found in Appendix B. Since the ℓ_p spaces are often treated in the main part of this book as special L_p spaces, Appendix C contains a development of the ℓ_p spaces from more basic principles for the reader not versed in the general theory of L_p spaces.

A Few Notes on the General Approach

The basic terminology and notation in this book are close to that used by Rudin in [200], with some extensions that follow the notation of Lindenstrauss and Tzafriri from [156] and [157].

Though most of the results in this text are divided in the usual way into lemmas, propositions, and theorems, a result that is really a theorem occasionally masquerades as an example. For instance, the theorems of Nikodým and Day that $(L_p[0,1])^* = \{0\}$ when $0 \le p < 1$ appear as Example 2.2.24, since such an L_p space is an example of a Hausdorff topological vector space whose dual does not separate the points of the space.

The theory in this book is developed for normed spaces over both the real and complex scalar fields. When a result holds for incomplete normed spaces as well as Banach spaces, the result is usually stated and proved in the more general form so that the reader will know where completeness is truly essential. However, results that can be extended from Banach spaces or arbitrary normed spaces to larger classes of topological vector spaces usually do not get the more general treatment unless the extension has a specific application to Banach space theory.

Any extensive treatment of the theory of normed spaces does require the study of two vector topologies that are not in general even metrizable, namely, the weak and weak* topologies. Much of the sequential topological intuition developed in the study of normed spaces can be extended to nonmetrizable vector topologies through the use of nets, so nets play a major role in many of the topological arguments given in this book. Since the reader might not be familiar with these objects, an extensive development of the theory of nets is given in the first section of Chapter 2.

This book is sprinkled liberally with examples, both to show the theory at work and to illustrate why certain hypotheses in theorems are necessary.

This book is also sprinkled liberally with historical notes and citations of original sources, with special attention given to mentioning dates within the body of the text so that the reader can get a feeling for the time frame within which the different parts of Banach space theory evolved. In ascribing credit for various results, I relied both on my own reading of the literature and on a number of other standard references to point me to the original sources. Among those standard references, I would particularly like to mention the excellent "Notes and Remarks" sections of Dunford and Schwartz's book *Linear Operators, Part I* [67], most of which are credited to Bob Bartle in the introduction to that work. Anyone interested in the rich history of this subject should read those sections in their entirety. I hasten to add that any error in attribution in the book you are holding is entirely mine.

In many cases, no citation of a source is given for a definition or result, especially when a result is very basic or a definition evolved in such a way that it is difficult to decide who should receive credit for it, as is the case for the definition of the norm function. It must be emphasized that *in no case does the lack of a citation imply any claim to priority on my part.*

The exercises, of which there are over 450, have several purposes. One obvious one is to provide the student with some practice in the use of the results developed in the text, and a few quite frankly have no reason for their existence beyond that. However, most do serve higher purposes. One is to extend the theory presented in the text. For example, Banach limits are defined and developed in Exercise 1.102. Another purpose of some of the exercises is to provide supplementary examples and counterexamples. Occasionally, an exercise presents an alternative development of a main result. For example, in Exercise 1.76 the reader is guided through Hahn's proof of the uniform boundedness principle, which is based on a gliding hump argument and does not use the Baire category theorem in any form. With the exception of a few extremely elementary facts presented in the first section of Chapter 1, none of the results stated and used in the body of the text have their proofs left as exercises. Very rarely, a portion of an example begun in the body of the text is finished in the exercises.

One final comment on the general approach involves the transliteration of Cyrillic names. I originally intended to use the modern scheme adopted by *Mathematical Reviews* in 1983. However, in the end I decided to write these names as the authors themselves did in papers published in Western languages, or as the names have commonly appeared in other sources. For example, the modern MR transliteration scheme would require that V. L. Šmulian's last name be written as Shmul'yan. However, Šmulian wrote many papers in Western languages, several of which are cited in this book, in which he gave his name the Czech diacritical transliteration that appears in this sentence. No doubt he was just following the custom of his time, but because of his own extensive use of the form Šmulian I have presented his name as he wrote it and would have recognized it.

Synopsis

Chapter 1 focuses on the metric theory of normed spaces. The first three sections present fundamental definitions and examples, as well as the most elementary properties of normed spaces such as the continuity of their vector space operations. The fourth section contains a short development of the most basic properties of bounded linear operators between normed spaces, including properties of normed space isomorphisms, which are then used to show that every finite-dimensional normed space is a Banach space.

The Baire category theorem for nonempty complete metric spaces is the subject of Section 1.5. This section is, in a sense, optional, since none of the results outside of optional sections of this book depend directly on it, though some such results do depend on a weak form of the Baire category theorem that will be mentioned in the next paragraph. However, this section has not been marked optional, since a student far enough along in his or her mathematical career to be reading this book should become familiar with Baire category. This section is placed just before the section on the open mapping theorem, closed graph theorem, and uniform boundedness principle for the benefit of the instructor wishing to substitute traditional Baire category proofs of those results for the ones given here.

Since a course in functional analysis is not a prerequisite for this book, the reader may not have seen the open mapping theorem, closed graph theorem, and uniform boundedness principle for Banach spaces. Section 1.6 is devoted to the development of those results. All three are based on a very specific and easily proved form of the Baire category theorem, presented in Section 1.3 as Theorem 1.3.14: Every closed, convex, absorbing subset of a Banach space includes a neighborhood of the origin.

In Section 1.7, the properties of quotient spaces formed from normed spaces are examined and the first isomorphism theorem for Banach spaces is proved: If T is a bounded linear operator from a Banach space X onto a Banach space Y, then Y and $X/\ker(T)$ are isomorphic as Banach spaces. Following a section devoted to direct sums of normed spaces, Section 1.9 presents the vector space and normed space versions of the Hahn-Banach extension theorem, along with their close relative, Helly's theorem for bounded linear functionals. The same section contains a development of Minkowski functionals and gives an example of how they are used to prove versions of the Hahn-Banach separation theorem. Section 1.10 introduces the dual space of a normed space, and has the characterizations up to isometric isomorphism of the duals of direct sums, quotient spaces, and subspaces of normed spaces. The next section discusses reflexivity and includes Pettis's theorem about the reflexivity of a closed subspace of a reflexive space and many of its consequences. Section 1.12, devoted to separability, includes the Banach-Mazur characterization of separable Banach spaces as isomorphs of quotient spaces of ℓ_1, and ends with the characterization of separable normed spaces as the normed spaces that are compactly generated so that the stage is set for the introduction of weakly compactly generated normed spaces in Section 2.8. This completes the basic material of Chapter 1.

The last section of Chapter 1, Section 1.13, is optional in the sense that none of the material in the rest of the book outside of other optional sections depends on it. This section contains a number of useful characterizations of reflexivity, including James's theorem. Some of the more basic of these are usually obtained as corollaries of the Eberlein-Šmulian theorem, but are included here since they can be proved fairly easily without it. The most important of these basic characterizations are repeated in Section 2.8 after the Eberlein-Šmulian theorem is proved, so this section can be skipped without fear of losing them. The heart of the section is a proof of the general case of James's theorem: A Banach space is reflexive if each bounded linear functional x^* on the space has the property that the supremum of $|x^*|$ on the closed unit ball of the space is attained somewhere on that ball. The proof given here is a detailed version of James's 1972 proof [117]. While the development leading up to the proof could be abbreviated slightly by delaying this section until the Eberlein-Šmulian theorem is available, there are two reasons for my not doing so. The first is that I wish to emphasize that the proof is really based only on the elementary metric theory of Banach spaces, not on arguments involving weak compactness, and the best way to do that is to give the proof before the weak topology has even been defined (though I do cheat a bit by defining weak sequential convergence without direct reference to the weak topology). The second is due to the reputation that James's theorem has acquired as being formidably deep. The proof is admittedly a bit intricate, but it is entirely elementary, not all that long, and contains some very nice ideas. By placing the proof as early as possible in this book, I hope to stress its elementary nature and dispel a bit of the notion that it is inaccessible.

Chapter 2 deals with the weak topology of a normed space and the weak* topology of its dual. The first section includes some topological preliminaries, but is devoted primarily to a fairly extensive development of the theory of nets, including characterizations of topological properties in terms of the accumulation and convergence of certain nets. Even a student with a solid first course in general topology may never have dealt with nets, so several examples are given to illustrate both their similarities to and differences from sequences. A motivation of the somewhat nonintuitive definition of a subnet is given, along with examples. The section includes a short discussion of topological groups, primarily to be able to obtain a characterization of relative compactness in topological groups in terms of the accumulation of nets that does not always hold in arbitrary topological spaces. Ultranets are not discussed in this section, since they are not really needed in the rest of this book, but a brief discussion of ultranets is given in Appendix D for use by the instructor who wishes to show how ultranets can be used to simplify certain compactness arguments.

Section 2.2 presents the basic properties of topological vector spaces and locally convex spaces needed for a study of the weak and weak* topologies. The section includes a brief introduction to the dual space of a topological vector space, and presents the versions of the Hahn-Banach separation theorem due to Mazur and Eidelheit as well as the consequences for locally convex spaces of Mazur's separation theorem that parallel the consequences for normed spaces of the normed space version of the Hahn-Banach extension theorem.

This is followed by a section on metrizable vector topologies. This section is marked optional since the topologies of main interest in this book are either induced by a norm or not compatible with any metric whatever. An F-space is defined in this section to be a topological vector space whose topology is compatible with a complete metric, without the requirement that the metric be invariant. Included is Victor Klee's result that every invariant metric inducing a topologically complete topology on a group is in fact a complete metric, which has the straightforward consequence that every F-space, as defined in this section, actually has its topology induced by a complete *invariant* metric, and thereby answers a question of Banach.

The versions of the open mapping theorem, closed graph theorem, and uniform boundedness principle valid for F-spaces are given in this section.

Section 2.4 develops the properties of topologies induced by families of functions, with special emphasis on the topology induced on a vector space X by a subspace of the vector space of all linear functionals on X.

The study of the weak topology of a normed space begins in earnest in Section 2.5. This section is devoted primarily to summarizing and extending the fundamental properties of this topology already developed in more general settings earlier in this chapter, and exploring the connections between the weak and norm topologies. Included is Mazur's theorem that the closure and weak closure of a convex subset of a normed space are the same. Weak sequential completeness, Schur's property, and the Radon-Riesz property are studied briefly.

Section 2.6 introduces the weak* topology of the dual space of a normed space. The main results of this section are the Banach-Alaoglu theorem and Goldstine's theorem. This is followed by a section on the bounded weak* topology of the dual space of a normed space, with the major result of this section being the Krein-Šmulian theorem on weakly* closed convex sets: A convex subset C of the dual space X^* of a Banach space X is weakly* closed if and only if the intersection of C with every positive scalar multiple of the closed unit ball of X^* is weakly* closed.

Weak compactness is studied in Section 2.8. It was necessary to delay this section until after Sections 2.6 and 2.7 so that several results about the weak* topology would be available. The Eberlein-Šmulian theorem is obtained in this section, as is the result due to Krein and Šmulian that the closed convex hull of a weakly compact subset of a Banach space is itself weakly compact. The corresponding theorem by Mazur on norm compactness is also obtained, since it is an easy consequence of the same lemma that contains the heart of the proof of the Krein-Šmulian result. A brief look is taken at weakly compactly generated normed spaces.

The goal of optional Section 2.9 is to obtain James's characterization of weakly compact subsets of a Banach space in terms of the behavior of bounded linear functionals. The section is relatively short since most of the work needed to obtain this result was done in the lemmas used to prove James's reflexivity theorem in Section 1.13.

The topic of Section 2.10 is extreme points of nonempty closed convex subsets of Hausdorff topological vector spaces. The Krein-Milman theorem is obtained, as is Milman's partial converse of that result.

Chapter 2 ends with an optional section on support points and subreflexivity. Included are the Bishop-Phelps theorems on the density of support points in the boundaries of closed convex subsets of Banach spaces and on the subreflexivity of every Banach space.

Chapter 3 contains a discussion of linear operators between normed spaces far more extensive than the brief introduction presented in Section 1.4. The first section of the chapter is devoted to adjoints of bounded linear operators between normed spaces. The second focuses on projections and complemented subspaces, and includes Whitley's short proof of Phillips's theorem that c_0 is not complemented in ℓ_∞.

Section 3.3 develops the elementary theory of Banach algebras and spectra, including the spectral radius formula, primarily to make this material available for the discussion of compact operators in the next section but also with an eye to the importance of this material in its own right.

Section 3.4 is about compact operators. Schauder's theorem relating the compactness of a bounded linear operator to that of its adjoint is presented, as is the characterization of operator compactness in terms of the bounded-weak*-to-norm continuity of the adjoint. Riesz's analysis of the spectrum of a compact operator is obtained, and the method used yields the result for real Banach spaces as well as complex ones. The Fredholm alternative is then obtained from this analysis. Much of the rest of the section is devoted to the approximation property, especially to Grothendieck's result that shows that the classical definition of the approximation property in terms of the approximability of compact operators by finite-rank operators is equivalent to the common modern definition in terms of the uniform approximability of the identity operator on compact sets by finite-rank operators. The section ends with a brief study of the relationship between Riesz's notion of operator compactness and Hilbert's property of complete continuity, and their equivalence for a linear operator whose domain is reflexive.

The final section of Chapter 3 is devoted to weakly compact operators. Gantmacher's theorem is obtained, as well as the equivalence of the weak compactness of a bounded linear operator to the weak*-to-weak continuity of its adjoint. The Dunford-Pettis property is examined briefly in this section.

The purpose of **Chapter 4** is to investigate Schauder bases for Banach spaces. The first section develops the elementary properties of Schauder bases and presents several classical examples, including Schauder's basis for $C[0,1]$ and the Haar basis for $L_p[0,1]$ when $1 \le p < \infty$. Monotone bases and the existence of basic sequences are covered, and the relationship between Schauder bases and the approximation property is discussed.

Unconditional bases are investigated in Section 4.2. Results are presented about equivalently renorming Banach spaces with unconditional bases to be Banach algebras and Banach lattices. It is shown that neither the classical Schauder basis for $C[0,1]$ nor the Haar basis for $L_1[0,1]$ is unconditional.

Section 4.3 is devoted to the notion of equivalent bases and applications to finding isomorphic copies of Banach spaces inside other Banach spaces. Characterizations of the standard unit vector bases for c_0 and ℓ_1 are given. Weakly unconditionally Cauchy series are examined, and the Orlicz-Pettis theorem and Bessaga-Pełczyński selection principle are obtained.

The properties of the sequence of coordinate functionals for a Schauder basis are taken up in Section 4.4, and shrinking and boundedly complete bases are studied. The final section of Chapter 4 is optional and is devoted to an investigation of James's space J, which was the first example of a nonreflexive Banach space isometrically isomorphic to its second dual.

Chapter 5 focuses on various forms of rotundity, also called strict convexity, and smoothness. The first section of the chapter is devoted to characterizations of rotundity, its fundamental properties, and examples, including one due to Klee that shows that rotundity is not always inherited by quotient spaces. The next section treats uniform rotundity, and includes the Milman-Pettis theorem as well as Clarkson's theorem that the L_p spaces such that $1 < p < \infty$ are uniformly rotund. Section 5.3 is devoted to generalizations of uniform rotundity, and discusses local uniform rotundity, weak uniform rotundity, weak* uniform rotundity, weak local uniform rotundity, strong rotundity, and midpoint local uniform rotundity, as well as the relationships between these properties.

The second half of Chapter 5 deals with smoothness. Simple smoothness is taken up in Section 5.4, in which the property is defined in terms of the uniqueness of support hyperplanes for the closed unit ball at points of the unit sphere and then characterized by the Gateaux differentiability of the norm and in several other ways. The partial duality between rotundity and smoothness is examined, and other important properties of smoothness are developed. Uniform smoothness is the subject of the next section, in which the property is defined using the modulus of smoothness and characterized in terms of the uniform Fréchet differentiability of the norm. The complete duality between uniform smoothness and uniform rotundity is proved. Fréchet smoothness and uniform Gateaux smoothness are examined in the final section of the chapter, and Šmulian's results on the duality between these properties and various generalizations of uniform rotundity are obtained.

Appendix A includes an extended description of the prerequisites for reading this book, along with a very detailed list of the changes that must be made to the presentation in Chapter 1 if this book is to be used for an undergraduate topics course in Banach space theory. **Appendices B** and **C** are included to support such a topics course. They are, respectively, a list of the properties of metric spaces that should be familiar to a student in such a course and a development of ℓ_p spaces from basic principles of analysis that does not depend on the theory of L_p spaces. **Appendix D** is a discussion of ultranets that supplements the material on nets in Section 2.1.

Dependences

No material in any nonoptional section of this book depends on material in any optional section, with the exception of a few exercises in which the dependence is clearly indicated. Where an optional section depends on other optional sections, that dependence is stated clearly at the beginning of the section.

The material in the nonoptional sections of Chapters 1 through 3 is meant to be taken up in the order presented, and each such section should be considered to depend on every other nonoptional section that precedes it. One important exception is that, as has already been mentioned, Section 1.5 can be omitted, since its results are used only in optional Section 2.3. However, the reader unfamiliar with the Baire category theorem will not want to skip this material.

All of the nonoptional sections of Chapters 1 and 2 should be covered before taking up Chapters 4 and 5. Chapter 4 also depends on the first two sections of Chapter 3. If the small amount of material in Chapter 4 on the approximation property is not to be skipped, then the development of that property in Section 3.4 must also be covered.

Some results about adjoint operators from Section 3.1 are used in Example 5.4.13. Except for this, Chapter 5 does not depend on the material in Chapters 3 and 4.

Appendix A does not depend on any other part of this book, except where it refers to changes that must be made to the presentation in Chapter 1 for an undergraduate topics course. Appendices B and C do not use material from any other portion of this book. Appendix D depends on Section 2.1 but on no other part of the book.

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